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Nonlinear-electron-response impact on the evolution of ion-acoustic wave packets in a magnetized plasma

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It has been argued by a number of authors that the development of intermittent, localized ion fluctuations in a two-species electron-ion drifting unmagnetized plasma is possible. Compressive ion-acoustic solitons can develop in such a plasma, but not rarefactive ones. In the case of a strongly magnetized two-component plasma where intermittent electrostatic ion-acoustic and ion-cyclotron wave packets can be excited, we show that stationary shock solutions can exist. Their existence is due to the electron nonlinearity rather than the ion nonlinearity as is the case when ion nonlinearities are included, as is often referred to as Korteweg–de Vries solitons. We also show that there is no stationary solution when the Zakharov-Kuznetsov boundary conditions are imposed. We analyze the stability of the shock solution and show that it is possible to trigger a nonlinear instability in the case where a rarefactive pulse is chosen as an initial condition. A nonlinear criterion for growth is developed.

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I. INTRODUCTION

Over the past decade, there have been a large number of observations of organized flow in fluids and plasmas commonly described as turbulent. Historically, turbulent flow has been characterized by extreme incoherence or randomness, the most successful theoretical treatments assuming a quasi-Gaussian probability distribution of the excited fluctuations. However, it has been observed that isolated coherent structures develop in some turbulent plasmas and fluids. Perhaps the most coherent structure known is the soliton, which maintains its characteristics even after colliding with other solitons. An example of intermittent behavior in plasma turbulence is given by the satellite measurements of the electric fields in the auroral zone [1–3], which show localized pulses propagating parallel to the magnetic field. On the other hand, numerical simulations have been performed to model specific experiments, and even replace laboratory experiments in some cases. The results of the simulations have also revealed the existence of intermittent fluctuations.

A major problem in determining the extent to which the turbulent state of some plasmas consists of coherent features is quantifying the idea of "intermittency." Intermittency has, strictly speaking, no canonical definition. A signal is called intermittent if it is subject to infrequent variations of large amplitude. While most observations of coherence use flow visualization in distinguishing incoherent from coherent, a mathematical technique that consists of constructing the probability distribution $\mathcal{P}(\delta f)$ of a field variable δf is often used.

Conventional theories of strong turbulence [4], such as the "direct-interaction approximation (DIA)" or the "eddy-damped quasinormal Markovian (EDQNM)" for example, would assume a quasi-Gaussian distribution, while the coherent-structure probability distribution would have non-Gaussian features. In other words, the probability distribution may have a long tail or non-Gaussian features that express the fact that there is a finite probability for observing a fluctuation with a large amplitude δf at any phase-space point (x,v). In other words, the conventional theories of strong turbulence fail to describe the observed intermittency in turbulent plasmas.

Sato and Okuda [5] were apparently the first to observe these intermittent ion fluctuations in numerically simulated plasmas; however, they did not provide as much detailed phase-space information as Barnes, Hudson, and Lotko [6]. In some of these simulations, ion-acoustic waves were linearly unstable, and in some others they were stable yet the intermittent structures developed. This suggests that nonlinear effects can play a major role in predicting the formation of coherent structures.

Several theoretical interpretations of the numerical simulation results of intermittent ion fluctuations described above have been proposed.

The conventional linear theories, namely, the quasilinear theories [7,8], predict that little free energy would be available to the fluctuations to grow to the large amplitudes observed in the numerical simulation results and the satellite observations. Quasilinear "plateauing" of the electron distribution and trapping arguments grossly underestimate the saturation amplitude when the spectrum consists of densely packed fluctuations. The nature of the limitation depends on the ratio of the autocorrelation time $\tau_{ac}=[k\Delta(\omega/k)]^{-1}$ to the electron trapping time $\tau_{tr}^{e}=(k\Delta v_e)^{-1}$ (where k^{-1} is the characteristic length or scale size of the ion fluctuation, Δv_e the electron velocity trapping width, and $\Delta(\omega/k)=(\omega/k)_{max}-(\omega/k)_{min}$ is the width of the phase velocity spectrum). According to quasilinear theory, which is valid when $\tau_{ac}<\tau_{tr}^{e}$, growth will cease for ion-acoustic waves when a quasilinear plateau of the electron distribution function f_{0e} has formed. In the opposite case, namely, when $\tau_{ac}>\tau_{tr}^{e}$, the Manheimer picture [8] of fluctuation growth predicts that growth of ion-acoustic waves will cease when the growth rate γ is equal to the electron bounce frequency ($k\Delta v_e$). In either case, the final saturation amplitude $e\Phi/T_e$ is very small (less than m_e/m_i). The limitations of the quasilinear and the Manheimer theories reside in their assumption of a homogeneous distribution of wave energy in space, i.e., the excited fluctuations are supposed to be "closely packed," while it is clear that the observed fluctuations are intermittently distributed. In other words, nonlinear phenomena and intermittency become important, and therefore the conventional theories mentioned, namely, the quasilinear and the Manheimer theories, fail to give a consistent picture of the physical phenomena.

The crucial importance of fluctuation intermittency should be emphasized at this point since it is a major factor in determining the amount of free energy available to the intermittent fluctuations for growth. The electrons, being the source of free energy in this problem, must not lose their energy and momentum by becoming trapped between two successive localized ion fluctuations. The ion fluctuations must be distant enough such that the electrons look like free particles carrying energy and momentum and transmitting it to the isolated ion fluctuations. What is meant by free electrons is that the electrons interacting with a typical fluctuation have not interacted with a spatially distant fluctuation (a threshold distance is obtained in the one dimensional case in Refs. [9,10]). The growing fluctuations then compete with each other for the relatively small amount of momentum when the electron to ion mass ratio $m_e/m_i \ll 1$. The large-amplitude fluctuations have an advantage and grow selectively. This state described by Dupree [9-11] and Hamza [12] as the "survival of the fittest" means that the system evolves to a state of isolated, large-amplitude fluctuations. In other words, the system evolves towards an intermittent state of fluctuations.

The relevance of intermittency is not limited to the one-dimensional problem already investigated, but rather expands to more complex problems where coherent structures have been observed to form and to last for a long time. A good example is shown by the satellite measurements [1,3] of the electric fields in the auroral zone. However, the purpose here is to explain the dynamical process that leads to the formation of these coherent fluctuations.

More specifically, we are interested in the early stages of the plasma in which the ion-acoustic wave is linearly unstable. The observations [1], along with the numerical simulations [13], have revealed wavelike fluctuations early on when the fluctuation amplitude is low and the ion motion is clearly linear. These fluctuations ultimately grow to very large amplitude, trap ions, and turn into Bernstein-Greene-Kruskal (BGK) equilibria [14] as shown in the results of the numerical simulations reported by Barnes, Hudson, and Lotko [6]. This later phase of the simulation and observation seems to be well understood, see for example Dupree [10] for the one-dimensional case. Dupree [9,10,16] proposed a theoretical explanation of these localized self-bound fluctuations. The proposed one-dimensional model suggests that random or turbulent fluctuations of the phase-space density of velocity space dimension Δv for which the potential energy $e\Phi$ is negative and of the order of $m(\Delta v)^2/2$ tend to form into self-trapped structures. These structures when isolated from each other are Berstein-Green-Kruskal [14] equilibria. For fluctuation speeds of the order of, or less than, the thermal speed v_{th}, the self-trapped structures take the form of depressions or "holes" in the local phase-space density. Generally speaking, turbulent fluctuations cannot be exact BGK modes since they are continually interacting or colliding with each other. The concept of a hole in the phase-space density such as a BGK mode was discussed by Berk, Nielson, and Roberts [16]. They investigated such holes analytically and numerically. They also discussed a useful gravitational analogy in which the holes may be regarded as gravitating masses. Infeld and Rolands [17] have discussed the stability of BGK equilibria in both magnetized and unmagnetized, fully three-dimensional, plasmas. The main question we propose to address here is the generalization of the one-dimensional results, concerning the nature of early time isolated low-amplitude fluctuations, to the three-dimensional case of a magnetized plasma.

In the case of ion-acoustic type of fluctuation some authors have tried, in the one-dimensional case, to identify the localized, long-lived fluctuations with nonlinear localized fluctuations such as Korteweg–de Vries (Kd V) solitons.

Some authors [13] have suggested that the observed localized fluctuations, in the one-dimensional case, are Korteweg–de Vries solitons. But we find, in the satellite data as well as in the simulation results, that the speed of the observed fluctuations is too slow and their potential (negative potential) of the wrong sign to be the conventional compressive (positive potential) ion acoustic Kd V soliton. In addition the compressive Kd V soliton being a positive potential structure would not reflect electrons which are the source of energy and momentum in this problem. Moreover, the Kd V soliton propagates with a velocity that depends on its amplitude, while the results of the simulations and the satellite measurements indicate that there is no dependence of the propagation velocity of the localized fluctuations on their amplitude. In addition, the localization of the compressive Kd V soliton is achieved as ion nonlinearities (quadratic in the electrostatic potential) balance wave dispersion. However, the simulation results [6] show that the localized ion fluctuations develop early on when the ion dynamics is still linear.

In an attempt to compare the different theoretical models concerning solitary waves and double layers in the auroral plasma to the actual observations Malkki *et al.* [18] come to the conclusion that the nonlinear ion hole model (see Tetrault [19] and references therein) is in best agreement with the observations both in its assumptions and predictions. The ion hole model is a one-dimensional model and needs to be extended to higher dimensions. The object of this paper is not to study the ion hole and its dynamics, but rather to identify the dynamic process that leads to the formation of such a coherent structure. Dupress and Hamza [20] have addressed this problem in the one-dimensional case, and the present work addresses the same question in the case of a strongly magnetized plasma.

In this paper we are primarily concerned with extending the results of the one-dimensional problem treated by Dupree and Hamza [20] to the more complex case of a magnetized plasma. The principal result of this paper is that when the linear growth rate exceeds the combined dispersion rates in both the parallel and perpendicular directions to the magnetic field, an electron nonlinearity (as contrasted to an ion nonlinearity in the conventional case) will counteract the effects of dispersion early on and allow the possibility of formation of localized fluctuations.

The one-dimensional results discussed by Dupree and Hamza [20] can be extended to the case of three-dimensional ion-acoustic wave packets excited in a strongly magnetized two-species electron-ion drifting plasma. Although the problem is more complex, the physics is similar. The excited three-dimensional ion-acoustic wave packets propagate along the magnetic-field lines and disperse in both the parallel and perpendicular directions relative to the magnetic field. The electrons being strongly magnetized travel along the field lines like beads on wires, and make their momentum available to the wave packets. In the case where there is very little dispersion in the perpendicular direction, the one-dimensional results hold, otherwise the rate of momentum input by the reflected and trapped electrons into the wave packet has to compensate the rate of momentum loss due to the decay of the maximum amplitude of the packet which now results from dispersion not only in the parallel direction but perpendicular direction as well. It is also important to note that when dispersing in the perpendicular direction the fluctuations have access to more electron momentum, namely, that of the electrons traveling along neighboring field lines.

It is also important to stress that the problem of the development of ion-acoustic solitons (compressive) in two-species magnetized plasmas was first approached by Zakharov and Kuznetsov [21] who derived an equation for the evolution of nonlinear ion-acoustic waves in magnetized plasmas (this equation will be referred to as the ZK equation from here on). The ZK equation describing weakly nonlinear ion-acoustic waves was derived using a fluid model and therefore does not take into account kinetic effects such as trapping or reflection of particles. The properties of the ZK equation have been investigated analytically as well as numerically by a number of authors, see for example Frycz and Infeld [22] and references therein or Murawski and Edwin [23] and references therein. It has been shown that the ZK equation is not integrable (i.e., cannot be derived from an integrable Hamiltonian) and does not admit an N-soliton solution, i.e., the solution to the ZK equation is a localized solution but not necessarily a soliton according to Zabusky's [24] definition of a soliton. More recently Lotko [13] have approached the problem of the development of localized nonlinear ion-acoustic waves in a magnetized plasma from the numerical point of view. They start their Vlasov simulations with a negative potential pulse and observe the formation of coherentlike structures. Finally Song *et al.* [25] showed numerically that it is possible to excite rarefactive localized solutions in a magnetized two-species plasma. As mentioned earlier, BGK equilibria qualify to explain the final stage of the simulation when the coherent, spatially localized fluctuations have fully developed and are assumed not to interact with one another. However, the sole purpose of the present work, as stressed once more, is to understand the scenario that eventually leads to the formation of these BGK equilibria.

In this paper we shall derive the equation governing the evolution of the excited ion fluctuations in a strongly magnetized plasma, where the ions are cold and linear. The equation consists of two parts. The first part is a linear Zakharov-Kuznetsov [21] equation without the quadratic nonlinearity due to the ions, and the second part consists of the nonlinear electron response. We show that when imposing the same boundary conditions as in the case of the ZK equation, the evolution equation does not have stationary solutions. However, when imposing different boundary conditions we find that a stationary shock solution can develop. The effects of the electron response on the fluctuations are analyzed thoroughly, and the growth rate for a nonlinear instability is derived in the case where the Zakharov-Kuznetsov boundary conditions are imposed.

Because of the complex nature of the problem, treating the problem exactly is just impossible. A "wave packet" has multiple peaks, and therefore makes the problem of investigating the effects of the nonlinear electron response on a multipeaked packet very difficult, because of the presence of not only reflected electrons but trapped electrons by the different peaks and wells seen by the electrons. In this paper we have elected to study the effects of the electrons on a single-peaked ion fluctuation. The results to be presented are only valid, consequently, for the case of a single-peaked ion fluctuation.

II. DERIVATION OF THE EVOLUTION EQUATION

A. The model

We now consider a strongly magnetized plasma (low $\beta \ll 1$), with $T_e \gg T_i$, where T_e and T_i represent the electron and ion temperatures, respectively. The equations that describe the magnetized system under consideration are the Vlasov and Poisson equations:

$$\frac{\partial f_j(\mathbf{x},\mathbf{v},t)}{\partial t} + \mathbf{v}\cdot\nabla f_j(\mathbf{x},\mathbf{v},t) + \left[-\frac{q_j}{m_j}\nabla\Phi + \mathbf{v}\times\mathbf{\Omega}_j \right]\cdot\nabla_{\mathbf{v}} f_j(\mathbf{x},\mathbf{v},t) = 0 \quad (1)$$

and

$$\nabla^2\Phi = -4\pi \sum_p q_j \int d\mathbf{v}\, f_j(\mathbf{x},\mathbf{v},t) \ , \quad (2)$$

where the subscript j in this case represents the plasma species.

Poisson's equation can be written more specifically, by specifying the different electron populations, namely, passing, trapped, and reflected electrons. The ions are considered to be cold and linear. Then Eq. (2) becomes

$$\nabla^2\Phi=4\pi e(n_e^P-n_i)+4\pi(n_e^T+n_e^R) , \quad (3)$$

where n_e^P, n_e^T, and n_e^R represent the "passing" or nonresonant electrons, the "trapped" and "reflected" electron charge densities, respectively. The charge densities can be written as

$$n_e^{P,T,R}=\int d\mathbf{v}\, f_e^{P,T,R}(\mathbf{x},\mathbf{v},t) ,$$

where the superscripts P, T, R stands for passing, trapped, and reflected electrons, respectively (purely nonlinear effects). It is important at this stage to clarify the validity of the assumption of linear cold ions. As mentioned earlier, we are primarily concerned with the dynamical process that leads to the development of localized structures in a two species magnetized plasma. To attain this we need to define the trapping times of electrons and ions as well as the dispersion rates in both the parallel and perpendicular directions.

To estimate the dispersion rates let us consider the case of a one dimensional pulse with an average group velocity $v_g=(\partial\omega/\partial k)|_{k=k_0}=\omega'(k_0)$. The spreading of the pulse can be accounted for by noting that a pulse with an initial spatial width Δx_0 must have inherent in it a spread of wave numbers $\Delta k\simeq 1/\Delta x_0$. This means that the group velocity, when evaluated for various k values within the pulse, has a spread in it of order

$$\Delta v_g\simeq\omega''(k)\Delta k\simeq\frac{2\pi\omega''(k)}{\Delta x_0} , \quad (4)$$

where ω'' represents the second derivative of the frequency with respect to the wave number. At time t this implies a spread in space of the order of $\Delta v_g t$. If we combine the uncertainties in position by taking the square root of the sum of squares, we obtain the width $\Delta x(t)$ at time t,

$$\Delta x(t)=\left[(\Delta x_0)^2+\left[\frac{2\pi\omega'' t}{\Delta x_0}\right]^2\right]^{1/2} . \quad (5)$$

Now that we have a qualitative picture about dispersion, we should estimate the time, τ_D, it takes the wave packet to undergo major changes, such as doubling its width and reducing its amplitude by a certain factor to be self-consistently determined:

$$\Delta x(\tau_D)=\frac{2\pi\omega''\tau_D}{\Delta x_0}=2\Delta x_0 . \quad (6)$$

For the one-dimensional problem of ion-acoustic waves this leads to

$$\tau_D\simeq\frac{\lambda^2}{\pi\omega''(k)}=\frac{1}{6\pi^2}\left[\frac{\lambda}{\lambda_{\mathrm{De}}}\right]^3\omega_{pi}^{-1} , \quad (7)$$

where λ, λ_{De}, and ω_{pi} represent the average wavelength, the electron Debye length, and the ion plasma frequency, respectively. This can be generalized to the three-dimensional case. Let us call $\tau_{D\parallel}$ and $\tau_{D\perp}$ the dispersion times in the parallel and perpendicular directions to the magnetic field, respectively. These two quantities are defined for, say, a wave packet with spectral widths $\Delta k_\parallel\simeq k_\parallel$ and $\Delta k_\perp\simeq k_\perp$ in the parallel and perpendicular directions, respectively, by

$$\Delta x_\perp(\tau_{D\perp})\simeq\Delta k_\perp\frac{\partial^2\omega_k}{\partial k_\perp^2}\tau_{D\perp}\simeq 2\Delta x_\perp(0) ,$$
$$\Delta x_\parallel(\tau_{D\parallel})\simeq\Delta k_\parallel\frac{\partial^2\omega_k}{\partial k_\parallel^2}\tau_{D\parallel}\simeq 2\Delta x_\parallel(0) . \quad (8)$$

In order to evaluate the dispersion times scales we need the linear dispersion relation, which allows us to obtain ω_k the eigenfrequency. To obtain the dielectric function we need to go back to Poisson's equation. The Fourier transform in space and time of Poisson's equation can be written as follows (linear ions, and nonlinear electrons):

$$k^2\epsilon_R(\mathbf{k},\omega)\Phi(\mathbf{k},\omega)=-\mathcal{O}_{\mathrm{FT}}\left(4\pi e\int_R d\mathbf{v}\,\tilde{f}_e\right) , \quad (9)$$

where $\mathcal{O}_{\mathrm{FT}}$ stands for Fourier transform in space and time, ϵ_R the real part of the dielectric function and $\tilde{f}_e$ is given by

$$\tilde{f}_e=f_e^{T,R}(\mathbf{x},\mathbf{v},t)-f_{0e}(\mathbf{v}) . \quad (10)$$

The real part of the dielectric function, the dispersion relation and other results of the linear analysis of the problem can be found in Davidson [7]. Expanding the real part of the dielectric function in equation (9) around the ion-acoustic branch leads to

$$k^2\left[\epsilon_R(\mathbf{k},\omega_{\mathbf{k}})+(\omega-\omega_{\mathbf{k}})\frac{\partial\epsilon_R(\mathbf{k},\omega)}{\partial\omega}\bigg|_{\omega=\omega_{\mathbf{k}}}\right]=-\mathcal{O}_{\mathrm{FT}}\left(4\pi e\int_R d\mathbf{v}\,\tilde{f}_e\right) , \quad (11)$$

where $\epsilon_R(\mathbf{k},\omega_{\mathbf{k}})=0$ is the linear dispersion relation determining $\omega_{\mathbf{k}}$, given by

$$\omega_{\mathbf{k}}=k_\parallel c_s[1-\tfrac{1}{2}k_\parallel^2\lambda_D^2-\tfrac{1}{2}k_\perp^2(\lambda_D^2+\rho_s^2)] . \quad (12)$$

This allows us to evaluate the dispersion time scales given by Eq. (8) for an ion-acoustic wave packet with $\Delta x_\parallel(0)\simeq\lambda_\parallel$ and $\Delta x_\perp(0)\simeq\lambda_\perp$, where $\lambda_\parallel$ and $\lambda_\perp$ are the average wavelengths in the parallel and perpendicular directions, respectively:

$$\tau_{D\perp}\simeq\frac{1}{2\pi^2}\left[\frac{\lambda_\parallel}{\lambda_{\mathrm{De}}}\right]^3\frac{\left[\dfrac{\lambda_\perp}{\lambda_\parallel}\right]^2}{\left[1+\dfrac{\rho_s^2}{\lambda_{\mathrm{De}}^2}\right]}\omega_{pi}^{-1} ,$$
$$\tau_{D\parallel}\simeq\frac{1}{6\pi^2}\left[\frac{\lambda_\parallel}{\lambda_{\mathrm{De}}}\right]^3\omega_\pi^{-1} , \quad (13)$$

where ρ_s is the ion gyrofrequency evaluated at the electron temperature. We should also point out that for ion-acoustic waves $k_\perp \ll k_\parallel$, i.e., $\lambda_\perp \gg \lambda_\parallel$. A typical value for the ratio of the average perpendicular wavelength to the average parallel wavelength in the auroral magnetosphere is about 6. This automatically makes the perpendicular dispersion time scale of an ion-acoustic wave packet propagating along a strong magnetic field much longer than the parallel dispersion time scale.

The next step consists of comparing the electron and ion trapping times to the dispersion times. Since the electrons are strongly magnetized, and therefore assumed to be one dimensional, their trapping time, τ_{TR}^e, can be defined as

$$\tau_{tr}^e=\left[\frac{m_e}{m_i}\right]^{1/2}\tau_{tr}^i\simeq\frac{\lambda_\parallel}{\left[\frac{-2e\Phi_m}{m_e}\right]^{1/2}} . \quad (14)$$

Therefore it takes longer for the ions to become trapped along the field lines than electrons. Consequently, at the early stages of the development of the fluctuations the electron dynamics is much more important than the ions dynamics. For example, for a strongly magnetized plasma characterized by $\omega_{ce}/\omega_{pe}\simeq 10$ (see for example the results of the numerical simulations reported by Barnes, Hudson, and Lotho [6]) and for ion-acoustic wave packets with $\lambda_\parallel\simeq 10\lambda_{De}$ and $\lambda_\perp\simeq 30\lambda_{De}$, the condition for the electron nonlinearity to be significant translates into a condition on the amplitude of the wave packet that is for

$$\tau_{tr}^e \ll \tau_{D\parallel}, \tau_{D\perp}, \quad \text{leads to} \quad \frac{-e\Phi_m}{T_e} \gg 10^{-4} , \quad (15)$$

while the ions would require an amplitude (m_i/m_e) larger than that required for the nonlinear electron effects to become important. This, in other words, justifies our linear treatment of the ions in the early stages of the development of the ion fluctuations since a smaller fluctuation amplitude is required for the electron reflection and trapping to occur.

Let us now go back to Eq. (11) and investigate the effects of the nonlinear electrons (right-hand side) on the evolution of the ion-acoustic wave packet. The expanded Poisson's equation becomes

$$(\omega-\omega_{\mathbf{k}})\Phi(\mathbf{k},\omega) = \frac{1}{k^2 \left.\frac{\partial\epsilon_R(\mathbf{k},\omega)}{\partial\omega}\right|_{\omega=\omega_{\mathbf{k}}}}\mathcal{O}_{FT}\left(4\pi e\int_R d\mathbf{v}\,\tilde{f}_e\right) \quad (16)$$

using the ion-acoustic dispersion relation given by Eq. (12) and multiplying by $-i$ Eq. (16) leads to

$$-i\omega\Phi(\mathbf{k},\omega)+ik_\parallel c_s\Phi(\mathbf{k},\omega)+\frac{(i)^3}{2}k_\parallel^3 c_s\lambda_D^2+\frac{(i)^3}{2}k_\parallel k_\perp^2(\lambda_D^2+\rho_s^2)=-\frac{1}{k^2\left.\frac{\partial\epsilon_R(\mathbf{k},\omega)}{\partial\omega}\right|_{\omega=\omega_{\mathbf{k}}}}(\mathcal{O}_{FT})\left[4\pi e\int_R dv\,\tilde{f}_e\right] , \quad (17)$$

We now introduce the "inverse Fourier transformation" $\mathcal{O}_{FT}^{-1}$,

$$\mathcal{O}_{FT}^{-1}\equiv\int\frac{d\mathbf{k}}{(2\pi)^3}\int\frac{d\omega}{2\pi}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} , \quad (18)$$

we then apply it to Eq. (17) to obtain the equation governing the evolution of the electrostatic potential $\Phi(\mathbf{x},t)$,

$$\frac{\partial\Phi(\mathbf{x},t)}{\partial t}+c_s\frac{\partial\Phi(\mathbf{x},t)}{\partial z}+\frac{c_s\lambda_D^2}{2}\frac{\partial^3\Phi(\mathbf{x},t)}{\partial z^3}+\frac{c_s(\lambda_D^2+\rho_s^2)}{2}\frac{\partial^3\Phi(\mathbf{x},t)}{\partial z\partial x_\perp^2}=\mathcal{O}_{FT}^{-1}\left[-\frac{1}{k^2\left.\frac{\partial\epsilon_R(\mathbf{k},\omega)}{\partial\omega}\right|_{\omega_{\mathbf{k}}}}\mathcal{O}_{FT}\left(4\pi e\int_R d\mathbf{v}\,\tilde{f}_e\right)\right] . \quad (19)$$

We should note at this point that the electrons are strongly magnetized, and therefore their motion can be approximated as a one-dimensional motion along the magnetic-field lines (like beads on wires).

The left-hand side of Eq. (20) above represents the linearized version of the Zakharov-Kuznetsov equation [21] (a three-dimensional generalized Korteweg–de Vries equation). The right-hand side of the same equation represents the nonlinear response of the electrons, it can be evaluated using $k_\perp \ll k_\parallel$,

$$\frac{2}{k_\parallel\left.\frac{\partial\epsilon_R}{\partial\omega}\right|_{\omega_{\mathbf{k}}}}\simeq k_\parallel^2\lambda_D^2 c_s . \quad (20)$$

The evolution Eq. (20) then becomes

$$\left[\frac{\partial}{\partial t}+c_s\frac{\partial}{\partial z}\left[1+\frac{\lambda_D^2}{2}\left\{\frac{\partial^2}{\partial z^2}+(1+\alpha)\nabla_\perp^2\right\}\right]\right]\Phi=\frac{\lambda_D^2}{2}c_s\frac{\partial n_{eR}}{\partial z} , \quad (21)$$

where n_{eR} is to be calculated in the next section, and α is

$$\alpha = \left[\frac{\rho_s}{\lambda_D} \right]^2 .$$

It is clear from the derivation of the evolution Equation (21) that the nonlinearity is due to the electrons rather than the ions, which is not the case for the Zakharov-Kuznetsov equation. The latter has a spatially localized solution, that can be explained by the balancing of dispersion by the quadratic ion nonlinearity. In our case we shall show that Eq. (21) derived above does not have a stationary solution when the boundary conditions imposed on the Zakharov-Kuznetsov equation are also imposed on Eq. (21). However, if one imposes different boundary conditions then it is possible to show that the equation allows for a stationary double-layer structure to exist, i.e., a BGK equilibrium is possible as we shall see.

B. Derivation of the nonlinear electron response

We now derive the expression for the resonant electron charge density. We consider a particle orbit analysis in order to determine the reflected electron charge density. We recall that the distribution function satisfying the Vlasov equation is a function of the particle orbits, i.e.,

$$f_e(\mathbf{x},\mathbf{v},t) = f_{0e}(\mathbf{v}_0(\mathbf{x},\mathbf{v},t)) = f_{0e}(v_{\parallel 0}, \mathbf{v}_{\perp 0}) , \tag{22}$$

where the particle orbits are defined by

$$\frac{d\mathbf{x}}{dt} = \mathbf{v} , \quad \frac{d\mathbf{v}}{dt} = -\frac{q}{m}\nabla\Phi(\mathbf{x},t) + \mathbf{v}\times\mathbf{\Omega} . \tag{23}$$

These equations can be expressed in terms of the energy, namely,

$$\frac{dE}{dt} = -e\frac{\partial\Phi(\mathbf{x},t)}{\partial t} , \quad E = \frac{1}{2}m(v_\parallel^2 + v_\perp^2) - e\Phi , \tag{24}$$

integrating Eq. (24) for strongly magnetized electrons [see, for example, Kruskal [26], or Qian, Lotko, and Hudson [27] for a direct application (the electrons can be considered one-dimensional electrons moving along the magnetic-field lines $v_\perp = 0$] leads to

$$v_{\parallel 0} = \mathrm{sgn}(v_{\parallel 0}) \left[v_\parallel^2 - \frac{2e}{m_e}\Phi(\mathbf{x},t) - \frac{2}{m_e}\Delta E \right]^{1/2} , \tag{25}$$

where the potential at $\mathbf{x} = \mathbf{x}_0$ and $t = t_0$ is chosen to be zero, and ΔE is given by

$$\Delta E = -e\int_{t_0}^{t} \frac{\partial\Phi(\mathbf{x}(\mathbf{x}_0,\mathbf{v}_0,\tau),\tau)}{\partial\tau} d\tau . \tag{26}$$

We should note that the time variation of the potential is assumed to be a slow time variation (the electrons basically see a quasistationary potential structure, provided the electron trapping time is much shorter than the dispersion times scales). Therefore the expression for the reflected electron charge density (the electrons are assumed to reflect off a localized potential double-layer structure, a single-peaked potential) can be expressed as follows for $z < z_m$, where z_m represents the position of the minimum potential $\Phi_m(x_\perp, z_m, t)$ along the field lines

$$-4\pi e\int_0^{+\infty} dv_\perp v_\perp \int_{-s_m(\mathbf{x})}^{+s_m(\mathbf{x})} dv_\parallel [f_{0e}(\mathbf{v}_0) - f_{0e}(\mathbf{v})] , \tag{27}$$

where

$$s_m(\mathbf{x},t) = \left[-\frac{2e}{m_e}[\Phi_m(x_\perp,t) - \Phi(x,t)] \right]^{1/2} \tag{28}$$

and for $z > z_m$ the reflected electron density is

$$4\pi e\int_0^{+\infty} dv_\perp v_\perp \left[\int_{-s_M(\mathbf{x})}^{s_M(\mathbf{x})} - \int_{-s_m(\mathbf{x})}^{+s_m(\mathbf{x})} \right] dv_\parallel \times [f_{0e}(\mathbf{v}_0) - f_{0e}(\mathbf{v})] , \tag{29}$$

where s_M is given by

$$s_M(\mathbf{x},t) = \left[-\frac{2e}{m_e}[\Phi_M(x_\perp,t) - \Phi(\mathbf{x},t)] \right]^{1/2} ,$$

and where $\Phi_M(x_\perp,t)$ represent the finite potential of the double layer. It should be noted one more time that the time dependence of the potential is a slow time dependence.

For a Maxwellian electron distribution function drifting in the parallel direction the resonant electron charge density can be evaluated using steps similar to the ones used for the one-dimensional case [10]. In the three-dimensional case the perpendicular velocity integral can be evaluated straightforwardly, the parallel velocity integral gives the same expression as the one found in the one-dimensional case. Let us call $F_{0e}(v_\parallel)$ the electron distribution after evaluating the perpendicular velocity integral

$$F_{0e}(v_\parallel) = \int_0^{+\infty} dv_\perp v_\perp f_{0e}(\mathbf{v}) . \tag{30}$$

Therefore the expression for the resonant charge density becomes

$$-4\pi e\int_{-s_m(\mathbf{x})}^{s_m(\mathbf{x})} dv_\parallel [F_{0e}(v_{0\parallel}) - F_{0e}(v_\parallel)] , \tag{31}$$

expanding the distribution functions around the average parallel phase velocity u of the wave packet leads to

$$-4\pi e F'_{0e}(u)\int_{-s_m}^{s_m} dv_\parallel v_{0\parallel} , \tag{32}$$

where the expression for $v_{\parallel 0}$ is given by Eq. (25). The expression (32) is similar to the expression derived for the main peak of the one-dimensional wave-packet case [12]. It can be evaluated exactly and leads to

$$n_{eR}(z<z_m) = 2\omega_{pe}^2 F'_{0e}(u)\Phi_m(x_\perp,t)\left[[1-\varphi(\mathbf{x},t)]^{1/2} + \varphi(\mathbf{x},t)\ln\left[\frac{1+[1-\varphi(\mathbf{x},t)]^{1/2}}{[\varphi(\mathbf{x},t)]^{1/2}} \right] \right] . \tag{33}$$

This can be compressed as follows:

$$n_{eR}(z<z_m) = \beta_e \mathcal{F}_m(\varphi) , \tag{34}$$

and similarly for $z > z_m$ one has

$$n_{eR}(z > z_m) = -\beta_e[\mathcal{F}_m(\varphi) - \mathcal{F}_M(\varphi)] , \tag{35}$$

where

$$\mathcal{F}_m(\varphi) = \Phi_m(x_\perp, t) \left[[1-\varphi(\mathbf{x},t)]^{1/2} + \varphi(\mathbf{x},t) \ln \left[\frac{1+[1-\varphi(\mathbf{x},t)]^{1/2}}{[\varphi(\mathbf{x},t)]^{1/2}} \right] \right] \tag{36}$$

and

$$\varphi(\mathbf{x},t) = \frac{\Phi(\mathbf{x},t)}{\Phi_m(x_\perp,t)} .$$

The expression for $\mathcal{F}_M$ can be obtained by replacing Φ_m by Φ_M in Eq. (36).

In the next section we shall look for possible stationary solutions to the evolution Equation (21) for different sets of boundary conditions.

III. THE STATIONARY DOUBLE-LAYER SOLUTION

In this section we shall show that if one assumes the same boundary conditions imposed on the Zakharov-Kuznetsov equation, that allow the formation of a spatially localized potential structure, then the evolution equation does not have a stationary solution in our case. However, if one imposes boundary conditions that allow a double-layer structure, or shock, to form then it is possible to show analytically that a stationary solution can be obtained. First we start by looking for a stationary solution by going into a moving frame defined by

$$\zeta = \frac{z - c_s t}{\lambda_D} , \quad \xi = \frac{x_\perp}{(\lambda^2 + \rho_s^2)^{1/2}} , \quad t \to \omega_\pi t , \tag{37}$$

then Eq. (21) becomes

$$\frac{\partial \Phi}{\partial t} + \frac{\partial}{\partial \zeta} \{ \nabla^2 \Phi - n_{eR} \} = 0 , \tag{38}$$

where $\Phi - \Phi(\zeta, \xi, t)$ and $n_{eR}(\zeta, \xi, t)$. We now look for a stationary solution such that $\Phi = \Phi(\zeta - \lambda t, \xi)$,

$$\frac{\partial}{\partial t} \equiv -\lambda \frac{\partial}{\partial \zeta} . \tag{39}$$

This therefore leads to rewriting Eq. (38) in the following form:

$$\frac{\partial}{\partial \zeta} (\nabla^2 \Phi - \lambda \Phi - \beta_e n_{eR}) = 0 , \tag{40}$$

thus integrating over ζ leads to a nonlinear Poisson's equation

$$\nabla^2 \Phi - \lambda \Phi - n_{eR} = F(\xi) , \tag{41}$$

where $F(\xi)$ is an arbitrary function to be determined by the boundary conditions. Depending on these boundary conditions Eq. (41) may or may not have a solution.

A. The Zakharov-Kuznetsov case

Zakharov and Kuznetsov were able to find a stationary, spatially localized solution to a generalized Korteweg–de Vries equation. The equation obtained included the quadratic nonlinearity due to the ions, but they did not include any of the electron nonlinearities. The final equation they derived is very similar to Eq. (41),

$$\nabla^2 \Phi - (\lambda - \Phi) \Phi = 0 .$$

They were then able to find a spherically symmetric solution; a three-dimensional localized solution which according to Zabuskki's definition of a soliton cannot be qualified as such since the equation is not integrable.

In our case the nonlinearity is due to electrons and happens to be more complicated than the quadratic nonlinearity that one gets when deriving a KdV-like equation. In this case we assume the following boundary conditions:

$$\begin{aligned} &|\zeta| \to \infty , \quad \Phi \to 0 , \quad \Phi_\zeta \to 0 , \quad \Phi_{\zeta\zeta} \to 0 , \\ &|\xi| \to \infty , \quad \Phi \to 0 , \quad \Phi_\xi \to 0 , \quad \Phi_{\xi\xi} \to 0 . \end{aligned} \tag{42}$$

For $\Phi \to 0$ one can approximate n_{eR} for $\zeta \to -\infty$ and therefore obtain

$$\nabla^2 \Phi + \beta_e \Phi \ln |\Phi| \approx 0 , \tag{43}$$

because $\beta_e \propto F'_{0e} < 0$ and that we have derived the nonlinear electron response assuming a negative single-peaked ion fluctuation, the second term in Eq. (43) is negative. Consequently $\nabla^2 \Phi > 0$ and therefore the equation has no solution converging to zero. In other words, our assumption of a single-peaked solution, a double-layer solution, is violated, which leads us to conclude that Eq. (41) along with the boundary conditions imposed does not have a stationary localized solution.

We have just argued that when imposing the Zakharov-Kuznetsov boundary conditions the equation describing the evolution of the single-peaked potential fluctuation does not admit a stationary solution.

B. The double-layer case

Let us now impose the following boundary conditions:

$$\begin{aligned} &\zeta \to +\infty , \quad \Phi \to \Phi_M , \quad \Phi_\zeta \to 0 , \quad \Phi_{\zeta\zeta} \to 0 , \\ &\zeta \to -\infty , \quad \Phi \to 0 , \quad \Phi_\zeta \to 0 , \quad \Phi_{\zeta\zeta} \to 0 . \end{aligned} \tag{44}$$

We are basically looking at whether a solution associated with the pseudopotential V of the type shown in Fig. 1 is a stationary solution to the evolution equation for the potential. In this case the nonlinear electron response is given by Eqs. (34)–(36). In this case the equations satisfied by the potential are

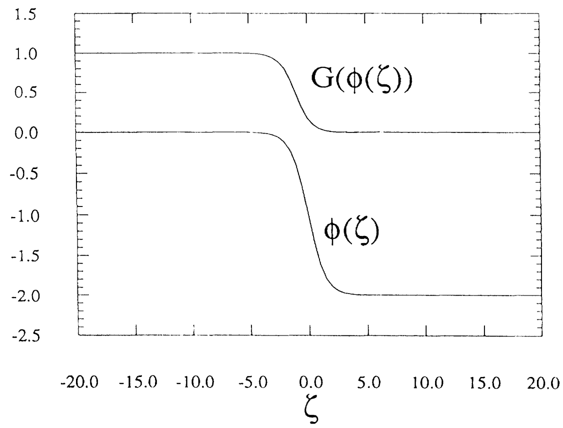

FIG. 1. The pseudopotential associated with a shock solution.

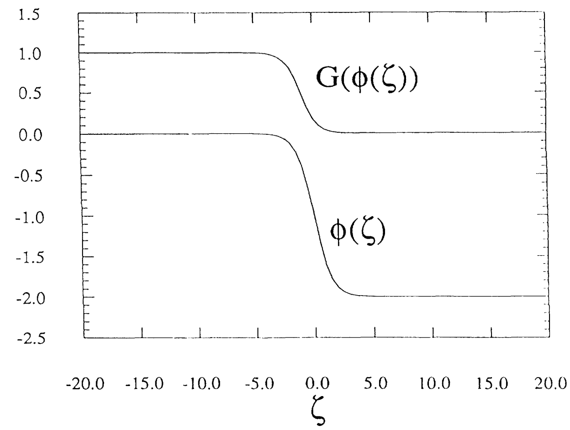

FIG. 2. The functional $G(\varphi)$ for a shock potential.

$$\nabla^2\Phi-\lambda\Phi-\beta_e\mathcal{F}_m(\varphi_m)=0\ , \tag{45}$$

$$\nabla^2\Phi-\lambda\Phi+\beta_e[\mathcal{F}_m(\varphi_m)-\mathcal{F}_M(\varphi_M)]=0\ , \tag{46}$$

where $\mathcal{F}_j$ is given by Eq. (36) and

$$\varphi_j(\mathbf{x},t)=\frac{\Phi(\mathbf{x},t)}{\Phi_j(x_\perp,t)}\ , \tag{47}$$

where Eq. (45) is satisfied for $\zeta<\zeta_m$ and Eq. (46) is satisfied for $\zeta>\zeta_m$.

The first equation can be approximated in the limit $\Phi\rightarrow 0$ to lead to

$$\nabla^2\Phi-\beta_e\Phi\ln\left|\frac{\Phi}{\Phi_m}\right|\approx 0 \tag{48}$$

and since $\beta_e<0$ it is consistent with the limit when $\zeta\rightarrow\infty$.

While the second equation can be approximated in the limit $\Phi\rightarrow\Phi_M$ by

$$\nabla^2\Phi-\lambda\Phi_M+\beta_e\mathcal{F}_m(\varphi_m^M)\approx 0\ , \tag{49}$$

where

$$\varphi_m^M=\frac{\Phi_M}{\Phi_m}\ . \tag{50}$$

Now in order for $\nabla^2\Phi$ to satisfy the boundary condition at $\zeta=\zeta_m$ we need

$$\nabla^2\Phi|_{\zeta_m}=\lambda\Phi_m=\lambda\Phi_M-\beta_e\mathcal{F}_m(\varphi_m^M)\ , \tag{51}$$

which leads to

$$\lambda\Phi_M-\lambda\Phi_m=\beta_e\mathcal{F}_m(\varphi_m^M)\ . \tag{52}$$

A trivial solution is $\varphi_m^M=1$. That is, the stationary solution is a shocklike solution. It is also clear that in this case electrons are only reflected on one side only [the nonlinear electron response vanishes on the other side since electrons are not reflected and expression (35) vanishes]. This solution, see Fig. 2, is achieved through the balancing of the dispersion term by the electron nonlinearity (as contrasted to the ion quadratic nonlinearity in the ZK equation).

IV. STABILITY ANALYSIS

It has been argued by Song *et al.* [25], from a numerical point of view, that rarefactive, spatially localized structures can develop in a two-species magnetized plasma. In order to obtain such a solution one has to treat the fully nonlinear problem, which is practically impossible to do analytically without having to introduce some kind of perturbation analysis. We, however, believe that it is the electron nonlinearity that needs to be included, and that dispersion is balanced by the nonlinear response of the electrons rather than the quadratic nonlinearity due to the ions as has been the case in many suggested models in the past. We have shown above that if one imposes the right boundary conditions then a shock solution becomes possible. We will show next, that even in the case where there are no stationary solutions, one can excite intermittently, spatially localized fluctuations.

Let us now consider the case where the Zakharov-Kuznetsov boundary conditions are imposed. We have shown earlier that in this specific case there is no stationary solution. Let us now investigate the effects of the nonlinear electron response on the development of ion fluctuations. The effects of nonlinear electrons on the evolution of one-dimensional ion-acoustic wave packets in a two-component unmagnetized plasma have already been considered in Hamza [12] and Dupree and Hamza [20], and it has been shown that it is possible to balance the dispersion by the electron nonlinearity to obtain a growing rarefactive solution. We shall show in this section that it is also possible to excite nonlinearly unstable shock solutions in a magnetized plasma. This in fact will allow us to understand the formation of strong double layers in auroral plasmas.

The equation (21) governing the evolution of the ion fluctuations was derived earlier. The rest of this section is going to be dedicated to the time-dependent solution to Eq. (21). The evolution can be written in the following form after introducing the normalized variables defined by Eq. (37), then Eq. (21) reduces to Eq. (38) which can be rewritten for convenience as

$$\frac{\partial \Phi}{\partial t}+\frac{\partial}{\partial \zeta}\{\nabla^2\Phi-n_{eR}\}=0 \ .$$

The solution to the linear equation is a linear ion-acoustic wave packet

$$\Phi_L(\zeta,\xi,t)=\frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}dk_{\parallel}\int_{-\infty}^{+\infty}dk_{\perp}\Phi(\mathbf{k})e^{(ik_{\parallel}\zeta+ik_{\perp}\xi+ik_{\parallel}k^2t)} \ ,$$

where

$$\Phi(\mathbf{k})=\int_{-\infty}^{+\infty}d\zeta\int_{-\infty}^{+\infty}d\xi\,\Phi(\zeta,\xi,0)e^{ik_{\parallel}\eta+ik_{\perp}\xi} \ .$$

One of the main assumptions is to assume a negative pulse for the initial condition. For an initial Gaussian pulse the integral over $k_\perp$ can be performed analytically while the integral over $k_\parallel$ is done numerically,

$$\Phi_L(\zeta,\xi,t)=\frac{1}{2\sqrt{\pi}}\int_{-\infty}^{+\infty}\frac{dk_{\parallel}}{(1+16k_{\parallel}^2t_{\perp}^2)^{1/4}}\exp\left[-\frac{k_{\parallel}^2}{4}-\frac{\xi^2}{1+k_{\parallel}^2t_{\perp}^2}\right]$$
$$\times\cos\left[k_{\parallel}\zeta+k_{\parallel}^3t_{\parallel}+\frac{1}{2}\arctan(4k_{\parallel}t_{\perp})-\frac{4k_{\parallel}t_{\perp}\xi^2}{1+16k_{\parallel}^2t_{\perp}^2}\right] \ ,$$

where $t_\parallel$ and $t_\perp$ are defined as follows for an initial condition with parallel and perpendicular widths $\lambda_\parallel$ and $\lambda_\perp$, respectively:

$$t_{\parallel}=\frac{t}{\lambda_{\parallel}^3} \ , \quad t_{\perp}=\frac{t}{\lambda_{\parallel}\lambda_{\perp}^2} \ .$$

The results are shown in Figs. (3) and (4) for two extreme cases. It is clear that the signature of the linear solution is an Airy function in the parallel direction. We should point out that the electrons are moving along the magnetic-field lines like beads on wires. When such wave packets are excited, the electrons start exchanging energy and momentum with the packets, and it is important to stress the fact that as the wave packets disperse in the perpendicular direction more electrons get reflected by the relative minima of the packets, and therefore provide a source of free energy and momentum to the packets. This wave-particle interaction can allow, as we shall see very soon, a nonlinear instability to set up. However, the main object of this section is to study the evolution of a single-peaked initial condition and the stability of the stationary solutions when they exist. In order to do so, let us multiply Eq. (53) and integrate over ζ, with the Zakharov-Kuznetsov boundary conditions imposed. This leads to

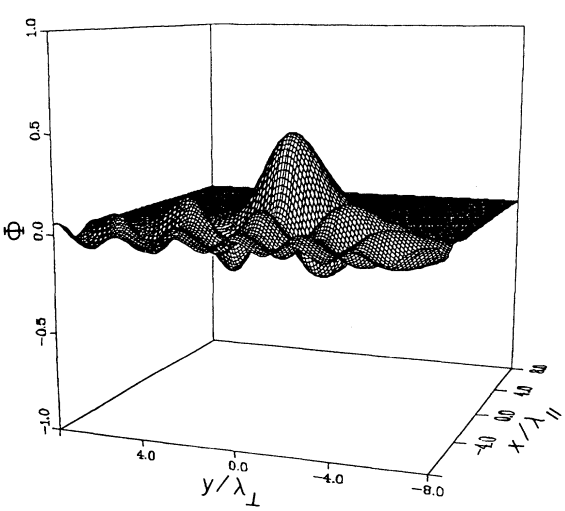

FIG. 3. The linear solution for a narrow initial condition (Gaussian), with parallel and perpendicular widths $\lambda_\parallel/\lambda_D=10$ and $\lambda_\perp/\lambda_D=5$, respectively, in units of Debye lengths, at time $\tau=\omega_{pi}t=300$ in units of ion plasma frequency.

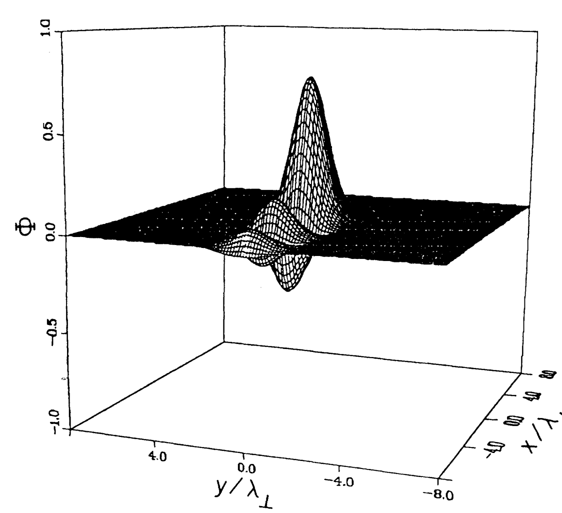

FIG. 4. The linear solution for a broad initial condition (Gaussian), with parallel and perpendicular widths $\lambda_\parallel/\lambda_D=10$ and $\lambda_\perp/\lambda_D=30$, respectively, in units of Debye lengths, at two time $\tau=\omega_{pi}t=300$ in units of ion plasma frequency.

$$\frac{\partial}{\partial t}\int_{-\infty}^{+\infty}d\zeta\frac{\Phi^2}{2}=\int_{-\infty}^{+\infty}d\zeta\frac{\partial\Phi}{\partial\zeta}(\nabla^2\Phi-\beta_e n_{eR}) . \quad (53)$$

This last equation can even be reduced by expressing explicitly the Laplacian to obtain

$$\frac{\partial}{\partial t}\int_{-\infty}^{+\infty}d\zeta\frac{\Phi^2}{2}=\int_{-\infty}^{+\infty}d\zeta\frac{\partial\Phi}{\partial\zeta}\left[\frac{\partial^2\Phi}{\partial\xi^2}-\beta_e n_{eR}\right] . \quad (54)$$

By assuming a negative potential as an initial condition one can write an explicit form for the electron nonlinearity, and then show that

$$\beta_e\frac{\partial\Phi}{\partial\zeta}n_{eR}=\frac{\partial\mathcal{N}_{eR}}{\partial\zeta} ,$$

then the second integral on the right-hand side becomes

$$\frac{\partial}{\partial t}\int_{-\infty}^{+\infty}d\zeta\frac{\Phi^2}{2}=\int_{-\infty}^{+\infty}d\zeta\frac{\partial\Phi}{\partial\zeta}\frac{\partial^2\Phi}{\partial\xi^2} -\mathcal{N}_{eR}(\zeta\to+\infty)+\mathcal{N}_{eR}(\zeta\to-\infty) . \quad (55)$$

Let us now integrate over ξ. The expression above becomes

$$\frac{\partial}{\partial t}\int_{-\infty}^{+\infty}d\zeta\int_{-\infty}^{+\infty}d\xi\frac{\Phi^2}{2}=\int_{-\infty}^{+\infty}d\zeta\int_{-\infty}^{+\infty}d\xi\frac{\partial\Phi}{\partial\zeta}\frac{\partial^2\Phi}{\partial\xi^2}-\int_{-\infty}^{+\infty}d\xi[\mathcal{N}_{eR}(\zeta\to+\infty)-\mathcal{N}_{eR}(\zeta\to-\infty)] . \quad (56)$$

It is clear that the first term on the right-hand side, due basically to dispersion in the perpendicular direction can be simplified by integrating by parts. Indeed, the dispersion terms spread the momentum in space but they are not sources of momentum and energy. Therefore, the only source of momentum is due to the nonlinear electron response. Equation (56) can now be written as follows:

$$\frac{\partial}{\partial t}\int_{-\infty}^{+\infty}d\zeta\int_{-\infty}^{+\infty}d\xi\frac{\Phi^2}{2}=-\int_{-\infty}^{+\infty}d\xi[\mathcal{N}_{eR}(\zeta\to+\infty)-\mathcal{N}_{eR}(\zeta\to-\infty)] +\frac{1}{2}\int_{-\infty}^{+\infty}d\zeta\left\{\left[\frac{\partial\Phi}{\partial\zeta}\frac{\partial\Phi}{\partial\xi}\right]-\left[\frac{\partial^2\Phi}{\partial\zeta\partial\xi}\Phi\right]\right\}_{\xi=-\infty}^{\xi=+\infty} . \quad (57)$$

This allows us to state that if the right-hand side of Eq. (57) is definite positive, then the system is nonlinearly unstable. The instability is purely due to the nonlinear electron response balancing the dispersion effects.

The next step is to evaluate $\mathcal{N}_{eR}$ and show that the system is nonlinearly unstable. The expression was given earlier for an initial negative potential peak,

$$\mathcal{N}_{eR}=\beta_e\mathrm{sgn}(\zeta_m-\zeta)\Phi_m^2(\xi,t)\left[(1-\varphi)^{3/2}+(1-\varphi)^{1/2}+\varphi^2\ln\left[\frac{\varphi^{1/2}}{1+(1-\varphi)^{1/2}}\right]\right] , \quad (58)$$

when substituted into the expression (57) it leads to

$$\frac{\partial}{\partial t}\int_{-\infty}^{+\infty}d\zeta\int_{-\infty}^{+\infty}d\xi\frac{\Phi^2}{2}=\frac{1}{2}\int_{-\infty}^{+\infty}d\zeta\left\{\left[\frac{\partial\Phi}{\partial\zeta}\frac{\partial\Phi}{\partial\xi}\right]-\left[\frac{\partial^2\Phi}{\partial\zeta\partial\xi}\Phi\right]\right\}_{\xi=-\infty}^{\xi=+\infty}+4\beta_e\int_{-\infty}^{+\infty}d\xi\,\Phi_m^2(\xi,t) . \quad (59)$$

It is clear that the right-hand side of Eq. (59) can be definitely positive and therefore the potential can grow in time due to the fact that the source of free energy provided by the nonlinear electrons can balance the dispersion effects. If we assume that when $\xi\to\pm\infty$ the potential and its ξ derivative vanish, then it becomes clear that the first term on the right-hand side of Eq. (59) vanishes and therefore a nonlinear instability is always present. It is also important to note that the one-dimensional problem is nonlinearly unstable (no ξ dependence).

Before treating the other case, let us investigate the case of a rarefactive pulse localized in the region bounded by ζ_1, ζ_2, $\xi_1(\zeta)$, and $\xi_2(\zeta)$ or vice versa, and with parallel and perpendicular widths $\lambda_\parallel$ and $\lambda_\perp$, respectively. The peak of the pulse being localized at ($\zeta=0$, $\xi=0$). The potential is assumed to vanish on this contour. The momentum balance equation in this case can be written in the following form:

$$\frac{\partial}{\partial t}\int_{\zeta_1}^{\zeta_2}d\zeta\int_{\xi_1(\zeta)}^{\xi_2(\zeta)}d\xi\frac{\Phi^2}{2}=\frac{1}{2}\int_{\zeta_1}^{\zeta_2}d\zeta\left\{\left[\frac{\partial\Phi}{\partial\zeta}\frac{\partial\Phi}{\partial\xi}\right]-\left[\frac{\partial^2\Phi}{\partial\zeta\partial\xi}\Phi\right]\right\}_{\xi_1(\zeta)}^{\xi_2(\zeta)}+\frac{1}{2}\int_{\xi_1}^{\xi_2}d\xi\left[\left[\frac{\partial\Phi}{\partial\zeta}\right]^2\right]_{\zeta_1(\xi)}^{\zeta_2(\xi)}+4\beta_e\int_{\xi_1}^{\xi_2}d\xi\,\Phi_m^2 . \quad (60)$$

In order to have marginal stability the right-hand side of Eq. (60) has to vanish. This leads to a marginal stability condition (after normalizing the variables ζ and ξ to $\zeta/\lambda_\parallel$ and $\xi/\lambda_\perp$)

$$\frac{\gamma_L\lambda_\parallel^3}{I+\frac{\lambda_\parallel^2}{\lambda_\perp^2}J}=1 , \quad (61)$$

where $\gamma_L \propto \beta_e F'_{0e}$ represents the linear ion-acoustic growth rate, and

$$I=\frac{\frac{1}{4}\int_{\xi_1}^{\xi_2}d\xi\left[\left[\frac{\partial\Phi}{\partial\zeta}\right]^2\right]_{\zeta_2(\xi)}^{\zeta_1(\xi)}}{\int_{\xi_1}^{\xi_2}d\xi\,\Phi_m^2}\,,$$

$$J=\frac{\frac{1}{4}\int_{\xi_1}^{\xi_2}d\zeta\left[\frac{\partial\Phi}{\partial\zeta}\frac{\partial\Phi}{\partial\xi}\right]_{\xi_2(\zeta)}^{\xi_1(\zeta)}}{\int_{\xi_1}^{\xi_2}d\xi\,\Phi_m^2}\,. \tag{62}$$

It is clear that the marginal stability condition for the one-dimensional case can be recovered by taking the limit when $\lambda\perp\to\infty$ to obtain $\gamma_L\lambda_\parallel^3=I=1$.

The shock solution case

Let us now analyze Eq. (21) in more detail, in the case where a stationary shock solution can develop. The equation can be written as

$$\frac{\partial\Phi}{\partial t}+\frac{1}{2}\frac{\partial}{\partial\zeta}\nabla^2\Phi=\frac{\partial R}{\partial\zeta}\,, \tag{63}$$

where $\Phi=\Phi(\zeta,\xi,t)$ and $R=\beta_e n_{eR}$. The second term on the left-hand side represents the dispersion, while the right-hand side represents the electron nonlinearity.

It is clear that in order to obtain a shock solution, the electron nonlinearity has to balance the dispersion. The momentum balance equation shows that if one imposes boundary conditions such as $\Phi\to 0$ as $\eta\pm\infty$, the system is nonlinearly unstable as shown and argued in the previous section. In the remaining of this paper we shall investigate the stability of the shock solution by perturbing around it.

We have shown earlier that indeed one can find a stationary solution to Eq. (63) by imposing proper boundary conditions. Next we will investigate thoroughly the stability problem by expanding around the stationary solution. But before getting into the detailed analysis, let us consider one moment equation, the momentum conservation equation obtained by multiplying Eq. (63) by Φ and integrating over space to obtain

$$\frac{\partial}{\partial t}\int_{-\infty}^{+\infty}d\zeta\int_{-\infty}^{+\infty}d\xi\frac{\Phi^2}{2}=\int_{-\infty}^{+\infty}d\xi\,\Phi_M N_{eR}(\Phi_M)-\int_{-\infty}^{+\infty}d\xi[\mathcal{N}_{eR}(\zeta\to+\infty)+\mathcal{N}_{eR}(\zeta\to-\infty)]$$
$$+\frac{1}{2}\int_{-\infty}^{+\infty}d\zeta\left\{\left[\frac{\partial\Phi}{\partial\zeta}\frac{\partial\Phi}{\partial\xi}\right]-\left[\frac{\partial^2\Phi}{\partial\zeta\,\partial\xi}\Phi\right]\right\}_{\xi=-\infty}^{\xi=+\infty}\,, \tag{64}$$

the boundary conditions being specified by Eq. (44). It is clear that this equation is different from Eq. (57) because of the different boundary conditions imposed. We should, however, note that $N_{eR}(\Phi_M)=0$ and therefore the momentum balance equation can be written as

$$\frac{\partial}{\partial t}\int_{-\infty}^{+\infty}d\zeta\int_{-\infty}^{+\infty}d\xi\frac{\Phi^2}{2}=\frac{1}{2}\int_{-\infty}^{+\infty}d\zeta\left\{\left[\frac{\partial\Phi}{\partial\zeta}\frac{\partial\Phi}{\partial\xi}\right]-\left[\frac{\partial^2\Phi}{\partial\zeta\,\partial\xi}\Phi\right]\right\}_{\xi=-\infty}^{\xi=-\infty}+2\beta_e\int_{-\infty}^{+\infty}d\xi\,\Phi_m^2(\xi,t)\,. \tag{65}$$

It is clear in this case as well that the system is unstable only if the electron nonlinearity overwhelms the dispersion effects. Again, if the dispersive effects are neglected, one automatically triggers a nonlinear plasma instability. This can be confirmed by expanding around the stationary shock solution as follows:

$$\Phi(\zeta,\xi,t)=\Phi_0(\zeta+\xi-t)+\epsilon\Phi_1(\zeta,\xi,t)\,. \tag{66}$$

The linearization of the evolution equation leads to the following equations:

$$\frac{\partial\Phi_0}{\partial t}+\frac{\partial}{\partial\zeta}[\nabla^2\Phi_0-R(\Phi_0)]=0 \tag{67}$$

and to first order in ϵ we obtain

$$\frac{\partial\Phi_1}{\partial t}+\frac{\partial}{\partial\zeta}\nabla^2\Phi_1-\beta_e\,\mathrm{sgn}(\zeta)F\left[\frac{\Phi_0}{\Phi_m}\right]\frac{\partial\Phi_1}{\partial\zeta}$$
$$-\beta_e\,\mathrm{sgn}(\zeta)G\left[\frac{\Phi_0}{\Phi_m}\right]\Phi_1=0\,. \tag{68}$$

The zeroth-order solution was derived in Sec. III. The functionals F and G are known, and are given by

$$F=\frac{1}{\sqrt{1-\varphi_0}}-\ln\left[\frac{1+\sqrt{1-\varphi_0}}{\sqrt{\varphi_0}}\right] \tag{69}$$

and

$$G=\frac{1}{2}\frac{\partial\varphi_0}{\partial\zeta}\left\{(1-\varphi_0)^{-3/2}+\frac{1}{(1-\varphi_0)^{1/2}+1-\varphi_0}+\frac{1}{\varphi_0}\right\}\,. \tag{70}$$

When neglecting the dispersion term in Eq. (68) we obtain an equation that can be solved using the method of characteristics. The equation is basically one dimensional, and enables us to explicitly look at the solution to the problem in the absence of dispersion,

$$\frac{d\zeta}{dt}=-\beta_e\,\mathrm{sgn}(\zeta)F\left[\frac{\Phi_0}{\Phi_m}\right] \tag{71}$$

and

$$\frac{d\Phi_1}{dt}=\beta_e\,\mathrm{sgn}(\zeta)G\left[\frac{\Phi_0}{\Phi_m}\right]\Phi_1\,, \tag{72}$$

where the d/dt is used to denote the directional derivative along the characteristics $C(\zeta,t)=\text{const}$, given by

$$\frac{\partial C}{\partial t}-\beta_e \text{sgn}(\zeta)F\left[\frac{\Phi_0}{\Phi_m}\right]\frac{\partial C}{\partial \zeta}=0 . \quad (73)$$

Equation (72) can be integrated to give

$$\Phi_1(C,t)=\Phi_1(C(\zeta,0),0) \times \exp(\beta_e \int_0^t d\tau \, \text{sgn}[\zeta(\tau)]G(\varphi_0(\zeta(\tau),\tau))) , \quad (74)$$

where again $\zeta(t)$ satisfies Eq. (71). The initial conditions are $\zeta(0)=\zeta_0$ and $\eta(t)=\eta$.

It is clear from Fig. (4) that for a shock solution Φ_0 the functional $G(\varphi_0)$ is always positive, and therefore for $\zeta(t)>0$ the perturbation Φ_1 grows in time, while it decays in time for $\zeta(t)<0$. The dispersion effects are crucial, and are indeed stabilizing, since dispersion basically spreads momentum in space.

On the other hand, we should note that by neglecting the dispersion effects one cannot obtain a stationary solution, since we expect the stationary solution to develop from background noise. Indeed, if the system is unstable, any fluctuation will be subject to both the electron nonlinearity and the dispersive effects, if one of these is neglected then the fluctuation keeps on growing or disperses.

V. SUMMARY AND CONCLUSION

The problem of ion-acoustic turbulence in both laboratory and space plasma physics has been debated for a long time. The leading theoretical models suggesting that turbulent fluctuations of soliton type can explain the different observations. However, it is clear that rarefactive ion-acoustic solitons cannot develop in a one-dimensional unmagnetized two-species plasma, and only compressive ones can form, and consequently one cannot explain the one-dimensional results of several numerical solutions. In a magnetized plasma, we have shown, when neglecting the ion quadratic nonlinearity, including the source of free energy in the electron, and imposing proper boundary conditions, that a stationary shock solution can form.

We have shown that if localized rarefactive fluctuations should develop then they are unstable and can grow to large amplitudes. We have analyzed the situation when ion-acoustic wave packets are excited. We have been able to show that the nonlinear electron response allows these linear wave packets to remain localized, and at the same time grow nonlinearly, and a criteria for nonlinear growth was derived. These results are a generalization of the one-dimensional case studied by Hamza (1989) and Dupree and Hamza (1990).

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